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# Wave equation migration with the phase-shift method

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Accurate methods for the solution of the migration of zero-offset seismic records have been developed. The numerical operations are defined in the frequency domain. The source and recorder positions are lowered by means of a phase shift, or a rotation of the phase angle of the Fourier coefficients. For applications with laterally invariant velocities, the equations governing the migration process are solved very accurately by the phase-shift method. The partial differential equations considered include the 15 degree equation, as well as higher order approximations to the exact migration process. The most accurate migration is accomplished by using the asymptotic equation, whose dispersion relation is the same as that of the full wave equation for downward propagating waves. These equations, however, do not account for the reflection and transmission effects, multiples, or evanescent waves. For comparable accuracy, the present approach to migration is expected to be computationally more efficient than finite-difference methods in general.

## INTRODUCTION

In recent years, migration methods based on the numerical solution of the wave equation have gained considerable acceptance. These migration techniques have their origin in the pioneering work of Claerbout. By defining the problem in a downward-moving coordinate system, Claerbout (1970, 1976) derived a simplified equation which lends itself to numerical treatment more conveniently than the full wave equation. This partial differential equation, which is often referred to as the 15 degree equation, has been solved so far by finite-difference methods.

More recently, Stolt (1978) used Fourier transform techniques for migration. Migration with Fourier transforms was also studied by Claerbout (1977) and Lynn (1977). In these studies, finite Fourier transforms are employed for obtaining a direct solution of the wave equation. Such direct solutions are known to exist for a large class of linear partial differential equations with constant coefficients. Thus, migration with these methods is limited to homogeneous media with constant-velocity function. In order to overcome this limitation, Stolt (1978) suggests coordinate transformations to cast the wave equation in velocity invariant form.

In this paper we develop solution methods for migration of seismic records in inhomogeneous media. This calls for the numerical solution of partial differential equations with variable coefficients. The numerical operations are defined in the frequency domain rather than in configuration space. The aim is to obtain the solution by operating on the Fourier coefficients of the seismic section. This permits us to formulate numerical procedures which are practically free from truncation errors (Gazdag, 1976). We find that the numerical algorithm formulated in the frequency domain is considerably simpler than any finite-difference method in configuration space. The reason for this is that the step-by-step process of lowering the source and the recorder positions is accomplished by implementing a phase change in the Fourier coefficients. Numerically, this is equivalent to a multiplication by a complex number of unit modulus. There is no stability condition associated with this operation. This means that the source and recorder positions can be lowered by any amount within one computational step.

In this paper we shall consider only migration of zero offset seismic data. In the following section, we set out the details of the phase shift method as applied

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to the 15 degree equation. The derivations apply to laterally homogeneous media. In the subsequent section, we extend this method to higher order equations. We obtain an asymptotic equation whose form is the same as the relativistic Schroedinger equation expressed in a downward-moving coordinate system. Next we attempt to generalize the phase shift method to laterally inhomogeneous media and discuss certain simplifications in the migration algorithm. The computations of the wave extrapolation and those of the inverse Fourier transformation with respect to the depth variable are combined into a single computational step. Finally, numerical results of migration examples are discussed.

## THE PHASE SHIFT METHOD; SECOND-ORDER EQUATION

The zero offset seismic section $p(x, t, \tau)$ may be considered as a wave field measured at some specified depth from the surface of the earth. The variables $x$, $t$, and $\tau$ are the horizontal position, the two-way traveltime, and the two-way vertical traveltime, respectively. Computationally, the migration process can be regarded as a numerical approximation of the changes in the wave field as the sources and the recorders are moved downward into the earth. The seismic section $p(x, t, \tau = 0)$ recorded at the surface serves as the initial condition for the solution $p(x, t, \tau)$, the seismic section which would have been observed, had the sources and the recorders been positioned at depth $\tau$. The subset $p(x, t = \tau, \tau)$ of all the computed seismic sections corresponds to the diffractor source distribution and provides the desired migrated section. This concept is illustrated by Claerbout (1976) in Figure (11-2-6), where the results are located along the diagonal of the $(z, t)$ grid or, according to the present notation, the $(\tau, t)$ grid, as suggested by Loewenthal et al (1976).

We shall demonstrate the phase-shift method with the equation

$$p_{t\tau} = -\frac{v^2}{8} p_{xx}. \qquad (1)$$

This is a second-order approximation to the two-dimensional scalar wave equation written in a downward-moving coordinate system (Claerbout, 1970, 1976). It is also known as the 15 degree equation.

In order to keep the details of the derivation simple and tractable, we shall assume no lateral velocity variations, i.e., $v = v(\tau)$. Let the finite Fourier transform of $p$ be defined as

$$P(k_x, \omega, \tau) = \frac{1}{4K\Omega} \sum\sum p(x, t, \tau) \cdot \exp[-i(k_x x + \omega t)], \qquad (2)$$

in which

$$K = \pi/\Delta x \quad \text{and} \quad \Omega = \pi/\Delta t, \qquad (3)$$

where $\Delta x$ and $\Delta t$ are the grid spacings. The summation in (2) is carried out for all frequencies $|k_x| \le K$ and $|\omega| \le \Omega$.

In view of definition (2), the partial differential equation (1) expressed in the frequency domain becomes

$$P_\tau = \frac{-i\, v^2 k_x^2}{8\omega} P. \qquad (4)$$

The solution to (4) can be written in the following form

$$P(\tau + \Delta\tau) = P(\tau) \exp(-i\,\phi\,\Delta\tau), \qquad (5)$$

in which

$$\phi = \frac{k_x^2}{8\,\omega\,\Delta\tau} \int_\tau^{\tau+\Delta\tau} v^2\, d\tau. \qquad (6)$$

If we define $v_{rms}$ as the root-mean-square value of the velocity averaged between the interval $\tau$ and $\tau + \Delta\tau$, i.e.,

$$v_{rms}^2 = \frac{1}{\Delta\tau} \int_\tau^{\tau+\Delta\tau} v^2\, d\tau, \qquad (7)$$

and let

$$m = 2\omega / v_{rms} \qquad (8)$$

for this interval of integration, then we can write $\phi$ in the simple form

$$\phi = \frac{\omega\, k_x^2}{2m^2}. \qquad (9)$$

Since solution (5) depends on the rms value of the velocity within any interval of integration, the velocity $v$ in (4) can be replaced by its rms value $v_{rms}$. With the help of (8), we can write (4) in a somewhat more convenient form,

$$P_\tau = \frac{-i\,\omega\, k_x^2}{2m^2} P. \qquad (10)$$

The desired migrated section is given by the subset $p(x, t = \tau, \tau)$ of all the computed seismic sections. Therefore, after each $\Delta\tau$ step, we compute

$$P(x, t = \tau, \tau) = \sum\sum P(k_x, \omega, \tau) \cdot \exp[i(k_x x + \omega\tau)]. \qquad (11)$$

It is emphasized that the phase shift $\phi\Delta\tau$ required for the extrapolation of a given wave component from $\tau$ to $\tau + \Delta\tau$ depends on the rms velocity across this layer of $\Delta\tau$ thickness.

Let $NX$ and $NT$ be the number of grid points along the $x$- and $t$-axes, respectively, over which $p(x, t, \tau)$ is defined. Then $P(k_x, \omega, \tau)$ has $NT \cdot NX/2$ complex data points. Approximately one-half of these Fourier coefficients can be set to zero, since those waves with

$$m^2 \leq k_x^2 \tag{12}$$

correspond to nonpropagating waves. One-half of the Fourier coefficients are nonphysical if $\Delta x = v_{\mathrm{rms}}\Delta\tau/2$. Normally, $\Delta x$ is greater than that, so the number of the deleted Fourier coefficients represent a smaller fraction than one-half of the total. However, if the maximum dip in the data is less than 90 degrees, the number of coefficients which need not be considered increases. Therefore, the assumption that only one-half of all the Fourier coefficients need to be included in the computation is a reasonable one. Thus, the computations in (5) require $NX \cdot NT/4$ complex multiplications. The computations in (11) for one $t$ value require approximately $NX \cdot NT/2$ additions as well as $NX \cdot NT/2$ multiplications. If $\Delta\tau \neq \Delta t$, for example, $\Delta\tau = r\Delta t$, then at each $\tau$ level (11) must be computed for $r$ different $t$ values. The total operation count for advancing one $\Delta\tau$ step [equation (5)] and reconverting the data into the $(x, t)$ domain [equation (11)] requires

$$(r + 2)NX \cdot NT/2 \quad \text{multiplications} \tag{13}$$

and

$$(r + 1)NX \cdot NT/2 \quad \text{additions.} \tag{14}$$

These figures do not include the computation of the complex multipliers and trigonometric functions, etc.

The remarkable advantage of solving (1) in the frequency domain is that the numerical integration of the Fourier transform (10) is reduced to a multiplication of $P$ by a complex number of unit modulus. This requires less computing than in finite-difference methods. This is seen immediately when we consider the $\Delta\tau = \Delta t$ case, i.e., when $r = 1$ in (13) and (14). This means that one $\Delta\tau$ step requires approximately 1.5 multiplications and only 1 addition per data point. On the other hand, a simple (explicit) finite-difference scheme for (1) would have to involve no less than six neighboring grid points. Such a finite-difference expression would require about 2 multiplications and 5 additions per grid point. If $\Delta\tau$ is much greater than $\Delta t$, i.e., $r \gg 1$, the operation count takes a turn in favor of the finite-difference method. In this case, however, the accuracy of migration with the finite-difference scheme cannot be expected to compare favorably with the accuracy of the phase-shift method. Moreover, the phase-shift method is equally suitable for solving higher-order equations to be discussed in the following sections, which could not be accomplished satisfactorily with finite-difference methods defined in terms of the variables $x$ and $t$. Another important property of the phase-shift method is accuracy. The numerical procedure is free of truncation errors. Moreover, there is no stability condition imposed on the magnitude of $\Delta\tau$.

Another type of error which deserves attention is related to truncation in the frequency-wavenumber domain. It is known that the highest frequency and wavenumber which can be represented on a grid of spacings $\Delta t$ and $\Delta x$ are given by $\pi/\Delta t$ and $\pi/\Delta x$, respectively. This upper limit is commonly referred to as the Nyquist frequency or the "folding" frequency. When a continuous signal, say $\exp(i\,\omega' t)$ with $\omega' > \pi/\Delta t$, is sampled at a rate $\Delta t$, then the digitized data cannot be distinguished from the digitized version of one of its aliases, e.g., $\exp[i(\omega' - 2\pi/\Delta t)t]$. We are concerned here about aliasing or folding errors resulting only from the numerical solution method of (1). Therefore, we shall assume that the grid spacings $\Delta t$ and $\Delta x$ are sufficiently small for representing the unmigrated section in all important detail. In other words, the section is specified completely by a finite number of Fourier coefficients associated with the computational grid. The question to be answered is this: Are there any Fourier modes produced by the migration process whose frequencies exceed the folding frequencies $\pi/\Delta t$ and $\pi/\Delta x$? From (11), we obtain migrated results for some depth $\tau$ from a set of Fourier coefficients, which were subjected to phase changes whose magnitude is proportional to the same variable $\tau$. This results in a frequency change in the migrated section corresponding to the dispersion relation of (1) (Claerbout, 1976),

$$k_\tau = \omega(1 - v^2k_x^2/8\omega^2), \tag{15}$$

where $k_\tau$ is the frequency (wavenumber) associated with the $\tau$ variable. From (15) and (12) we see that $k_\tau \leq \omega$. This implies that the migrated results contain no higher frequencies than the unmigrated section. Thus, if both sections (before and after migration) are represented on the same grid, then there are no apparent aliasing or "folding" errors associated with the migration process.

## FOURTH-ORDER AND ASYMPTOTIC EQUATIONS

Equation (1), whose solution we have considered so far, is characterized by the dispersion relation expressed in (15). This is a second-order approximation to the dispersion relation of the full wave equation, which is given by

$$k_\tau = \omega\left(1 - \frac{v^2 k_x^2}{4\omega^2}\right)^{1/2}. \tag{16}$$

The fourth-order approximation to (16) is

$$k_\tau = \omega\left(1 - \frac{v^2 k_x^2}{8\omega^2} - \frac{v^4 k_x^4}{128\omega^4}\right). \tag{17}$$

The partial differential equation with this dispersion relation is

$$p_{tt\tau} = -\frac{v^2}{8} p_{ttxx} - \frac{v^4}{128} p_{xxxx}. \tag{18}$$

This equation corresponds to equation (10-3-17) of Claerbout (1976). We note that $t$ represents the two-way traveltime, and that $\tau$ is the two-way vertical traveltime. The Fourier transform of (18) can be written as

$$P_\tau = \left[-i\omega\left(\frac{v^2 k_x^2}{8\omega^2} + \frac{v^4 k_x^4}{128\omega^4}\right)\right] P, \tag{19}$$

which corresponds to equation (10-3-17) of Claerbout (1976). The numerical solution of (19) is as described for the second-order case, except for the amount of phase shift. If the velocity $v$ can be regarded as some constant over the interval $(\tau, \tau + \Delta\tau)$, then $\phi$ is calculated for this layer from

$$\phi = \omega\left[\frac{v^2 k_x^2}{8\omega^2} + \frac{v^4 k_x^4}{128\omega^4}\right]. \tag{20}$$

However, if there are significant velocity variations within the interval under consideration, then the correct expression for $\phi$ is

$$\phi = \frac{\omega}{\Delta\tau}\int_\tau^{\tau+\Delta\tau}\left[\frac{v^2 k_x^2}{8\omega^2} + \frac{v^4 k_x^4}{128\omega^4}\right] d\tau. \tag{21}$$

The solution to (19) is expressed by (5) with $\phi$ given by (20) or (21).

If additional higher-order terms are used in computing $\phi$, the solution (5) becomes a higher-order approximation to that of the full wave equation. With the help of the Taylor series approximation to the square root, which is

$$(1 - z^2)^{1/2} = 1 - \frac{z^2}{2} - \frac{z^4}{8} - \frac{z^6}{16} - \cdots, \tag{22}$$

we can write the exact expression for $\phi$ in the following form

$$\phi = \omega\left[1 - \left(1 - \frac{v^2 k_x^2}{4\omega^2}\right)^{1/2}\right], \tag{23}$$

providing that $v$ is constant within the interval $(\tau, \tau + \Delta\tau)$ under consideration. If this is not the case, $\phi$ must be determined from the expression

$$\phi = \frac{\omega}{\Delta\tau}\int_\tau^{\tau+\Delta\tau}\left[1 - \left(1 - \frac{v^2 k_x^2}{4\omega^2}\right)^{1/2}\right] d\tau. \tag{24}$$

When (24) is used to compute the rate of phase change $\phi$, then (5) yields the numerical solution of the equation

$$P_\tau = i\omega\left[\left(1 - \frac{v^2 k_x^2}{4\omega^2}\right)^{1/2} - 1\right] P, \tag{25}$$

which is expressed in a downward-moving coordinate system. Equation (25), expressed in a stationary coordinate system, is

$$P_\tau = i\omega\left[1 - \frac{v^2 k_x^2}{4\omega^2}\right]^{1/2} P, \tag{26}$$

which is known as the relativistic Schroedinger equation (Claerbout, 1976, p. 202).

The solution to the asymptotic equation (25) is obtained from (5) when the correct phase (24) is used. These results are characterized by a dispersion relation which is very close to that of the full wave equation. Naturally, the exact representation of a section is limited in that only a finite number of waves can be represented over any computational grid. This limitation is true for any computational method. An advantage of the present approach to migration (over finite-difference methods) is that all the wave components which are represented on the grid are extrapolated correctly, without truncation errors.

The frequency domain representation of a seismic section implies periodic boundary conditions for the partial differential equations under consideration. Consequently, there is a possibility for some objects to migrate across the boundaries of the computational domain. Such phenomena produce incorrect results in the neighborhood of these boundaries. It is important to note, however, that migrated results near the boundaries are always unreliable, and most often incorrect, with no regard to the boundary conditions and the numerical method being used. This can be seen by considering that accurate migration results near the boundary would require information from both sides of that boundary, which is not available. The effects of periodicity on the migrated section can be eliminated in practice by choosing the computational domain somewhat larger than the actual seismic section and padding the extra space with zeros.

## LATERALLY INHOMOGENEOUS MEDIA

We have considered so far laterally homogeneous media only. In such cases the velocity is independent of the horizontal variable $x$. The numerical approaches described in this paper are particualrly suitable for this class of migration problems for two reasons. First, (25) and (26) represent a significantly better description of the migration process than lower-order equations do. The second reason is that the phase-shift method gives very accurate results within the limitations related to representing the seismic section as a finite set of double Fourier series, whose effects we have discussed above. When the medium is laterally inhomogeneous, we lose both of these important advantages. Therefore, a great deal of caution is required when dealing with lateral velocity variations. Notwithstanding these limitations, we can still expect to obtain practical results through simple modification of the phase-shift method based on judicious application of some physical principles.

Perhaps one of the most important examples of strong horizontal velocity variation is encountered at the ocean bottom which is inclined at some angle from the horizontal. In order to traverse such an interface, the source and recorder positions ($\tau$) must be advanced downward at different rates depending on their horizontal positions. In order to incorporate this $x$-dependence into the solution, $P(k_x, \omega, \tau)$ must be Fourier transformed with respect to $k_x$ in each $\Delta\tau$ step. Naturally, this requires additional computation. Alternatively, a different method (say finite-difference) could be used for traversing such critical regions. Even though the treatment of the ocean bottom requires special consideration, this extra cost is most likely offset by the very effective processing of data in reaching the bottom. It requires only one $\Delta\tau$ step for lowering the source and recorder positions down to any depth in the water itself.

One way to account for weak horizontal velocity variations is by undermigrating or overmigrating selectively, using some average velocity. Let us Fourier transform both sides of (1) with respect to the variable $t$ only, which gives

$$p_\tau = i\frac{v^2}{8\omega}p_{xx}, \tag{27}$$

whose solution can be expressed as

$$p(x, \omega, \tau+\Delta\tau) = p(x, \omega, \tau) + \int_\tau^{\tau+\Delta\tau} \frac{i\,v^2(x, \tau)}{8\omega} p_{xx}(x, \omega, \tau)\, d\tau. \tag{28}$$

The first-order approximation of this integral can be written as

$$p(x, \omega, \tau+\Delta\tau) \cong p(x, \omega, \tau) + \frac{i\,v^2(x, \tau)}{8\omega} p_{xx}(x, \omega, \tau)\,\Delta\tau. \tag{29}$$

Our goal is to remove the $x$-dependence from the velocity variable and to account for it in the variable representing the limit of the integration. We do this with the help of the lateral rms velocity $\bar{v}$, which is defined as

$$\bar{v}^2(\tau) = \frac{1}{X}\int_0^X v^2(x, \tau)\, dx, \tag{30}$$

in which $X$ is the horizontal length of the seismic section under consideration. Since we wish to use $\bar{v}$ as the migration velocity, rather than $v$, we define a new variable

$$\xi = \int_0^\tau (v^2/\bar{v}^2)\, d\tau, \tag{31}$$

which will also set the upper limit of the definite integral in conjunction with $\bar{v}$. Substituting the derivative of (31), $d\xi/d\tau = v^2/\bar{v}^2$, into (29) we obtain

$$p(x, \omega, \tau+\Delta\tau) \cong p(x, \omega, \tau) + \frac{i\,\bar{v}^2}{8\omega} p_{xx}(x, \omega, \tau)\,\Delta\xi. \tag{32}$$

In (32) the $v^2/\bar{v}^2$ ratio is absorbed in the increment $\Delta\xi = (d\xi/d\tau)\Delta\tau$, or equivalently, $\Delta\xi = (v^2/\bar{v}^2)\Delta\tau$. When (32) is written in the form of a definite integral, i.e., as

$$p(x, \omega, \tau+\Delta\tau) \cong p(x, \omega, \tau) + \int_\tau^{\tau+\Delta\xi} \frac{i\,\bar{v}^2}{8\omega} p_{xx}(x, \omega, \tau)\, d\tau, \tag{33}$$

the increment $\Delta\xi$ appears in the upper limit of the integral rather than $\Delta\tau$ as in the case of (28). If we take the lower limit of the integral as zero, i.e., $\tau = 0$, and the depth of interest expressed by the two-way traveltime as $\tau'$, i.e., $\tau' = \tau + \Delta\tau$ in (33), we obtain the expression

$$p(x, \omega, \tau') \cong p(x, \omega, 0) + \int_0^{\xi'} \frac{i\,\bar{v}^2}{8\omega} p_{xx}(x, \omega, \tau)\, d\tau, \tag{34}$$

in which the correspondence between $\xi'$ and $\tau'$ is expressed in (31). The physical significance of (34)

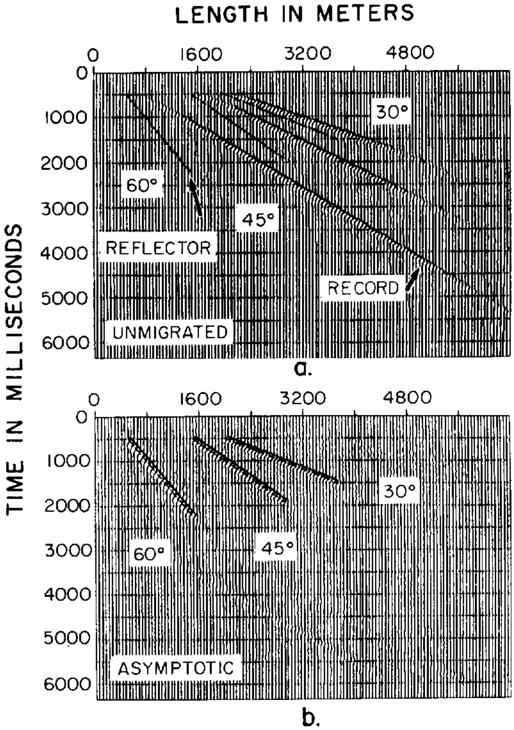


FIG. 1. (a) Synthetic zero-offset record section of three reflectors directed at 30, 45, and 60 degrees. (b) Migration of (a) using the asymptotic equation.

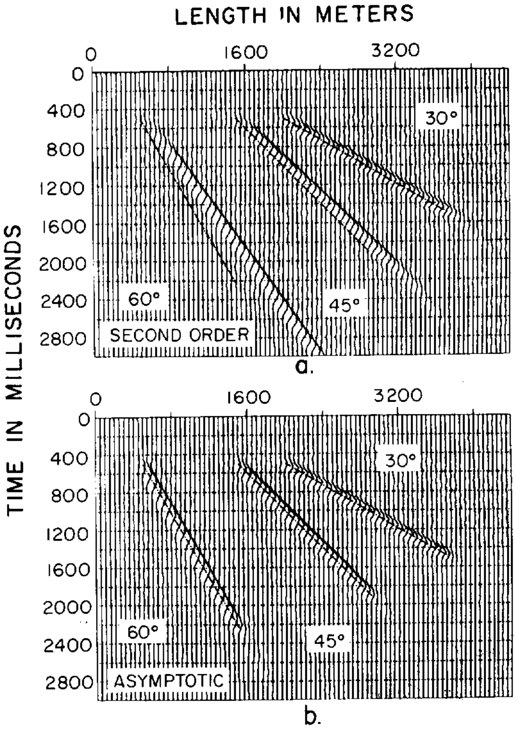


FIG. 2. Magnified views of the section shown in Figure 1a after migration using (a) the second-order equation, and (b) the asymptotic equation.

is that when $v^2$ is replaced by its lateral mean value $\overline{v}^2$, the migrated data at depth $\tau'$ is given by $p(x,\ t=\tau',\ \tau=\xi')$ rather than by $p(x,\ t=\tau',\ \tau=\tau')$ as before. The difference between $\xi'$ and $\tau'$ represents the amount of overmigration ($\xi' > \tau'$) or undermigration ($\xi' < \tau'$) required in order to account for the difference between $v^2$ and its lateral mean $\overline{v}^2$.

## PROGRAMMING CONSIDERATIONS FOR AN EFFICIENT ALGORITHM

As discussed above, the present approach to migration consists of (1) the extrapolation of the wave downward by operating on the Fourier coefficients, followed by (2) the inverse Fourier transformation of the correctly migrated data. These two operations expressed by (5) and (11) can be combined into a single operation. The solution to (4) can be expressed with reference to $P$ at $\tau = 0$, i.e.,

$$P(\tau) = P(0)\exp(-i\psi\tau), \tag{35}$$

in which

$$\psi = \frac{k_x^2 v_{\text{rms}}^2}{8\omega}, \tag{36}$$

where

$$v_{\text{rms}}^2 = \frac{1}{\tau}\int_0^\tau v^2\, d\tau. \tag{37}$$

By letting

$$m = 2\omega / v_{\text{rms}}, \tag{38}$$

we obtain

$$\psi = \frac{\omega k_x^2}{2m^2}. \tag{39}$$

Notice that (39) is the same expression as (9). The difference is in the definition of the rms velocity, (7) and (37). By substituting (35) into (11), we obtain

$$p(x,\ t=\tau,\ \tau) = \sum\sum P(k_x,\ \omega,\ 0)\cdot \cdot \exp\{i[k_x x + (\omega - \psi)\tau]\}. \tag{40}$$

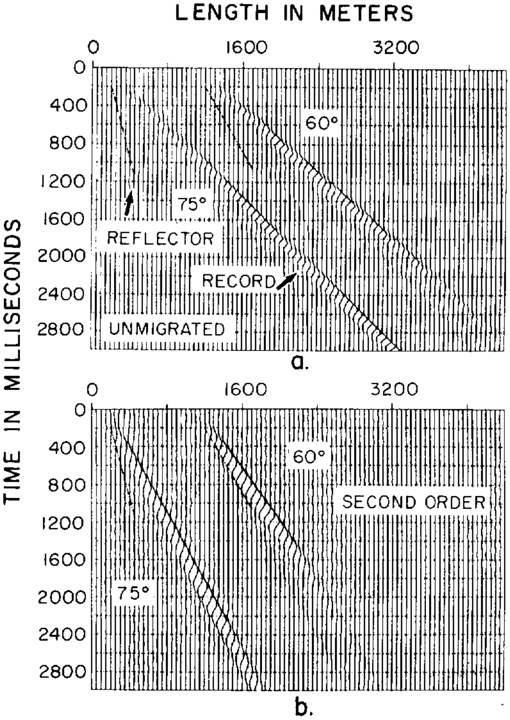


FIG. 3. (a) Unmigrated zero offset section of two reflectors inclined at 60 and 75 degrees. (b) Migration of (a) using the second-order equation.

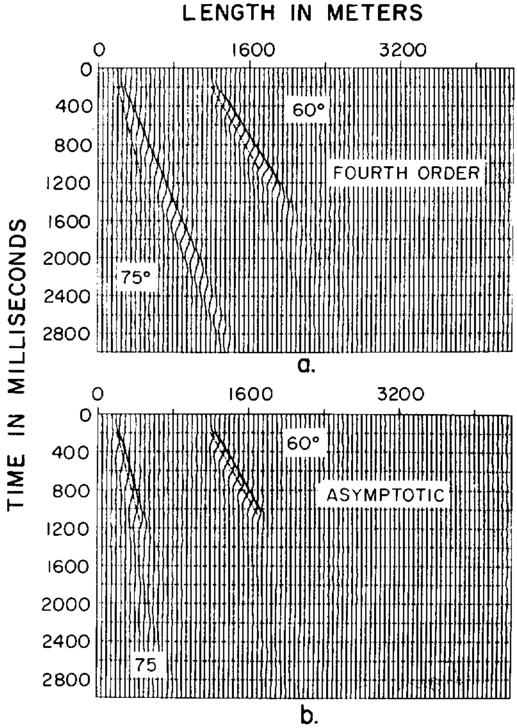


FIG. 4. Migration of the synthetic record section shown in Figure 3a using (a) the fourth-order equation, and (b) the asymptotic equation.

By making use of (39), we can write (40) as

$$p(x,\, t=\tau,\, \tau) = \sum\sum P(k_x,\, \omega,\, 0)\cdot \cdot \exp\left\{i\left[k_x x + \left(1 - \frac{k_x^2}{2m^2}\right)\omega\tau\right]\right\}. \quad (41)$$

Equations (40) and (41) give results which are solutions to the second-order approximations given by (1). Higher-order solutions are obtained if instead of (39), $\psi$ is computed from a higher-order approximation. In the case of the asymptotic equation, $\psi$ is given by

$$\psi = \frac{\omega}{\tau}\int_0^{\tau}\left[1 - \left(1 - \frac{v^2 k_x^2}{4\omega^2}\right)^{1/2}\right]d\tau, \quad (42)$$

whose substitution into (40) gives

$$p(x,\, t=\tau,\, \tau) = \sum\sum P(k_x,\, \omega,\, 0)\cdot \cdot \exp\{i[k_x x + \omega \cdot \cdot \int_0^{\tau}(1 - v^2 k_x^2/4\omega^2)^{1/2}\, d\tau]\}. \quad (43)$$

For each $\tau$ value, (41) or (43) represent approximately $NX \cdot NT/2$ additions and $NX \cdot NT/2$ multiplications, in addition to the operations required to compute the exponent itself. The computational complexity of (43) is roughly that of (11). However, in this direct approach the extra complex multiply (5) has been eliminated. Therefore, this direct method requires less than one-half of the operations necessary for the two-step procedure outlined previously.

In the special case when $v$ is constant, (43) becomes,

$$p(x,\, t=\tau,\, \tau) = \sum\sum P(k_x,\, \omega,\, 0)\cdot \cdot \exp[i(k_x x + k_\tau \tau)], \quad (44)$$

where $k_\tau$ is the frequency variable with respect to the variable $\tau$, given by

$$k_\tau = \omega\left[1 - \frac{v^2 k_x^2}{4\omega^2}\right]^{1/2}. \quad (45)$$

Equation (44) is essentially the same as equation (50) of Stolt (1978). We note again that $k_\tau \leq \omega$, which means that the problem can be solved, in principle at least, without errors due to aliasing.

## RESULTS

We tested the phase-shift method on synthetic zero-offset record sections. The record sections were generated by implementing the theory of Trorey (1970) for seismic diffractions. All numerical results were obtained by computing on a 128 × 128 grid, using $\Delta x = 50$ m, $\Delta t = \Delta \tau = 50$ msec. The migration velocity was constant, $v = 2000$ m/sec. When working with this velocity value, the two-way vertical traveltime $\tau$ measured in msec is numerically equivalent to the depth expressed in meters.

The migration examples represent results obtained from three different equations, which are solved numerically by the phase-shift method. The second-order approximation corresponds to the well-known 15 degree equation (1). Migration by means of (18) or (19) is called the fourth-order approximation. As the number of higher order terms (22) is increased without limit, the resulting approximation approaches equation (25) asymptotically. Therefore, migration by (25) is referred to as the asymptotic approximation. The asymptotic equation has a dispersion relation which is identical to the full wave equation for downward-traveling waves in constant-velocity media. Therefore, it is clearly superior to lower-order approximations. This is demonstrated by our results beyond any doubt, particularly for steeper dips.

For the correct interpretation of the figures, it is important to emphasize that the record sections were *not* subjected to any kind of smoothing before migration. A synthetic trace is a sequence of pulses of very short duration. The time at which these pulses occur is usually between two grid points $\Delta t$ apart. The pulse is represented on the grid by sharing its value between the two nearest grid points. Thus, the synthetic record contains a great deal of high-frequency components limited only by the Nyquist frequency. We found that we did not need to work with some kind of wavelets, whose high-frequency content is considerably less than that of a pulse of

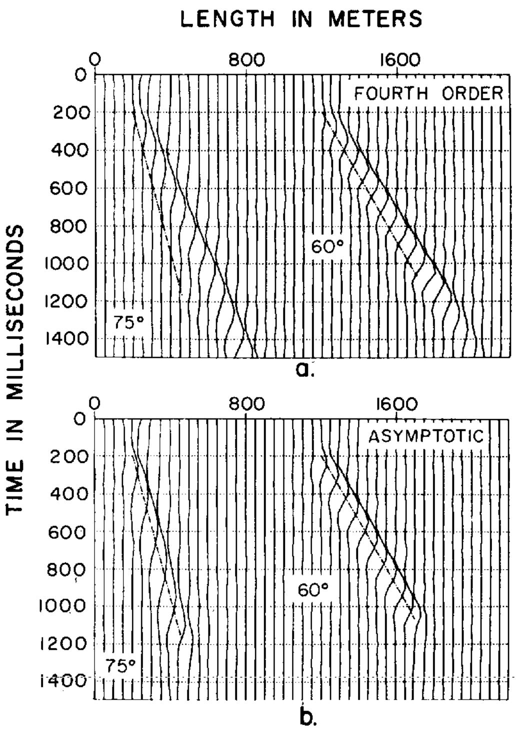


FIG. 5. Magnified views of the migrated sections shown in Figure 4.

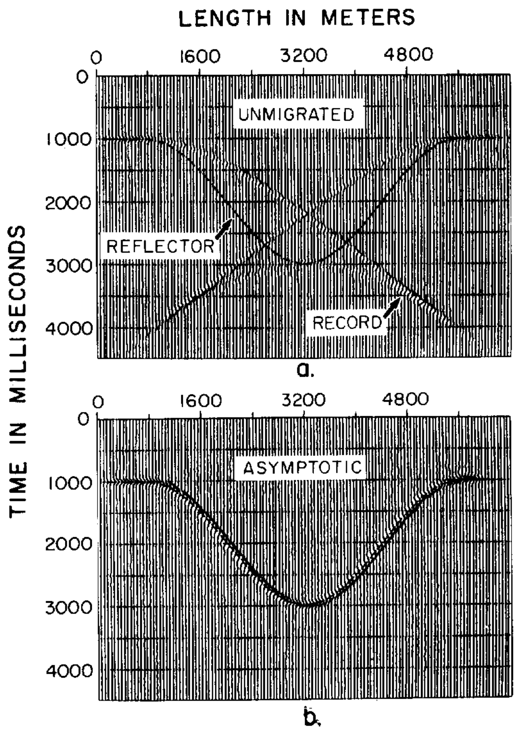


FIG. 6. (a) Synthetic zero-offset record section of a dipping reflector. (b) Migration of (a) using the asymptotic equation.

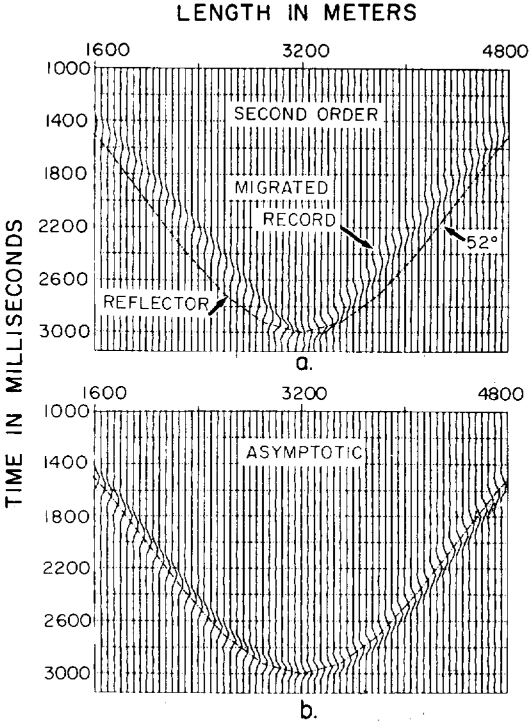


FIG. 7. Magnified views of the section shown in Figure 6 after migration using (a) the second-order equation, and (b) the asymptotic equation.

approximately $\Delta t$ width. This is the consequence of migrating all waves accurately.

Figure 1 shows the synthetic zero-offset record section of three plane reflectors and the migrated section using equation (25). The migrated section using the second-order approximation is shown in Figure 2a. It is remarkable that the "15 degree equation" results in a rather good migration of the reflector directed at 30 degrees. This leads us to believe that this equation might have been called the "30 degree equation" if accurate numerical methods had been used initially for solving it. While the quality of migration with the 15 degree equation deteriorates for reflectors with higher dips, all reflectors are migrated properly by the asymptotic expression (25) as shown in Figure 2b (magnified view of Figure 1b).

We have also experimented with reflectors up to 75 degrees. The record section and the migrated sections processed by the three different equations are shown in Figures 3, 4, and 5. The reflectors had to be short in order to keep the record section within the computational domain whose extent is 6400 × 6400 in units of m and msec. None of the lower-order approximations can migrate the 75 degree reflector with acceptable accuracy. With the asymptotic equation, however, we obtain good migration as shown in Figures 4b and 5b. This is quite remarkable if we consider that the 75 degree reflector spans only six traces.

We notice that the migrated records appear to spread out over several $\Delta t$ lengths. This could suggest some shortcoming of the numerical methods. This is not the case, however. The reason for the widening of the migrated records with increasing angle is related to the resolution of the unmigrated data. The synthetic record section defined on a grid can only represent data whose spread (thickness) is at least the grid size $\Delta t$ (or $\Delta x$). In practice, this thickness is usually wider than $\Delta t$ due to the sharing of the pulse between two neighboring grid points. When these data are migrated, even with the best possible method, it will retain this "thickness." Thus, what we can expect to obtain from the migration is a reflector whose thickness is roughly $2\Delta t$. The cross-section of such reflector along the $\tau$ axis is $(1/\cos\alpha)$ times wider. For $\alpha = 60$ and 75 degrees, this corresponds to $4\Delta t$ and $7.7\Delta t$, respectively. This is approximately what we observe in Figure 5b. This broadening in $\tau$ can also be viewed as the corollary to the dispersion relation (16). Since $k_\tau \leq \omega$, the frequency spectrum of the migrated section is narrower (in $k_\tau$) than that of the unmigrated section (in the variable $\omega$). The narrower frequency bandwidth implies broadening in $\tau$, which is in complete agreement with our observation. Figures 6 and 7 show migration results of a dipping reflector consisting of 18 plane segments. The reflector is a piecewise linear approximation to a sinusoid with maximum dip of 52 degrees. For this synthetic section, Figure 7 provides a comparison of the migration accuracy of the 15 degree equation and that of the asymptotic expression.

## CONCLUSIONS

We have described numerical methods for wave equation migration based on Fourier transform techniques. The algorithm is defined in the frequency domain, rather than in configuration space. The computations are simple, since the lowering of the source and recorder positions is implemented by incrementing the arguments of the complex Fourier coefficients. This amounts to a complex multiplication which is more economical than finite-difference methods. For laterally invariant migration velocities, the partial differential equations can be solved with high accuracy. The equations under consideration do not account for the reflection coefficients or evanescent waves. The most important result of this paper is that the phase-shift method can be used to solve the relativistic Schroedinger equation numerically, whose dispersion relation is identical to that of the wave equation for downward-traveling waves in constant velocity media. The phase shift method was tested on synthetic zero offset records, including reflectors with 75 degree dip, with excellent results.

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