3-D Elastic Wave Propagation for a VSP Geometry

D. Kessler and Dan Kosloff, Tel Aviv Univ., Israel*

Summary

We present a method for a solution of the three dimensional elastic wave equation for VSP geometry. This solution is operated on a three dimensional cylindrical grid using the multi-domain approach. Discretization of the wavefield is carried out on a grid of r θ and z , where r is the distance from the center, θ is the angular angle and z is depth. Spatial derivatives are performed using the Chebychev expansion along the radial direction, and using the Fourier expansion along the angular and the vertical direction.

We combine the equations of conservation of momentum with the stress-strain relation to yield a system of nine equations for the displacements and for the stresses. This system is resorted to a first order system which includes the variables that are needed for boundary conditions construction and for domain decomposition.

Boundary conditions and patching of grids are constructed by the use of the characteristic variables of the wave equation. The numerical algorithm is tested on the Cray YMP super computer.

Equations of Motion

The equations of motion we use are formed in cylindrical coordinates. We combine the equations of momentum conservation with the stress-strain relation for an isotropic elastic medium undergoing infinitesimal deformation. The momentum conservation equations are given by (Fung, 1965):

[1]

$$\rho\ddot{U}_r = \frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_{rr}\right) + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\partial\sigma_{rz}}{\partial z} - \frac{\sigma_{\theta\theta}}{r} + f_r$$

$$\rho\ddot{U}_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{r\theta}\right) + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{\theta z}}{\partial z} + f_\theta$$

$$\rho\ddot{U}_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_{rz}\right) + \frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta} + \frac{\partial\sigma_{zz}}{\partial z} + f_z$$

where U_r , U_θ and U_z are respectively the displacements in r , θ and z directions; σ_{rr} , $\sigma_{\theta\theta}$, σ_{zz} , $\sigma_{r\theta}$, σ_{rz} and $\sigma_{\theta z}$ are the stress components; f_r , f_θ and f_z are body forces per unit volume, and ρ denotes the density.

The stress-strain relations are:

[2]

$$\sigma_{rr} = \left(\lambda + 2\mu\right)\frac{\partial U_r}{\partial r} + \frac{\lambda}{r}\frac{\partial U_\theta}{\partial\theta} + \lambda\frac{\partial U_z}{\partial z} + \frac{\lambda}{r}U_r$$

$$\sigma_{\theta\theta} = \lambda\frac{\partial U_r}{\partial r} + \frac{\lambda + 2\mu}{r}\frac{\partial U_\theta}{\partial\theta} + \lambda\frac{\partial U_z}{\partial z} + \frac{\lambda + 2\mu}{r}U_r$$

$$\sigma_{zz} = \lambda\frac{\partial U_r}{\partial r} + \frac{\lambda}{r}\frac{\partial U_\theta}{\partial\theta} + \left(\lambda + 2\mu\right)\frac{\partial U_z}{\partial z} + \frac{\lambda}{r}U_r$$

$$\sigma_{r\theta} = \frac{\mu}{r}\frac{\partial U_r}{\partial\theta} + \mu\frac{\partial U_\theta}{\partial r} - \frac{\mu}{r}U_\theta$$

$$\sigma_{rz} = \mu\frac{\partial U_r}{\partial z} + \mu\frac{\partial U_z}{\partial r} \qquad \text{and}$$

$$\sigma_{\theta z} = \mu\frac{\partial U_\theta}{\partial z} + \frac{\mu}{r}\frac{\partial U_z}{\partial\theta}$$

. λ is the rigidity modulus and μ is the shear modulus.

Solution Scheme

For the treatment of the boundary conditions concurrent values of the variables $\dot{U}_r$, $\dot{U}_\theta$, $\dot{U}_z$, σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} , $\sigma_{r\theta}$, σ_{rz} and σ_{zz} are required.

Downloaded 05 Jun 2012 to 72.20.129.98. Redistribution subject to SEG license or copyright; see Terms of Use at http://segdl.org/

Thus we recast [1] and [2] to a system of nine coupled first order equations which are given by:

[3]

$$\frac{\partial}{\partial t}V = A\,\frac{\partial}{\partial r}V + B\,\frac{\partial}{\partial \theta}V + C\,\frac{\partial}{\partial z}V + U + W$$

where V is the variables vector

$$\left[\dot{U}_r,\ \dot{U}_\theta,\ \dot{U}_z,\ \sigma_{rr},\ \sigma_{\theta\theta},\ \sigma_{zz},\ \sigma_{r\theta},\ \sigma_{rz},\ \sigma_{\theta z}\right]^T,$$

A, B, and C are the 9×9 terms matrices which includes material parameters, U is the vector

$$\left[\frac{\sigma_{rr}-\sigma_{\theta\theta}}{\rho r},\ \frac{2\sigma_{r\theta}}{\rho r},\ \frac{\sigma_{rz}}{\rho r},\ \frac{\lambda}{r}U_r,\ \frac{\lambda+2\mu}{r}U_r,\ \frac{\lambda}{r}U_r,\ -\frac{\mu}{r}U_\theta\right]^T$$

and W is the vector which contains the source terms.

The numerical algorithm is based on solution of system [3]. Solution of the system [3] requires operation of spatial derivatives in r , θ and z directions, and a scheme for time integration. Along r direction we use the Chebychev expansion for the derivative:

[4]

$$\frac{\partial}{\partial r}F(r_j) = \sum_{i=0}^{N_r} b_k\, T_k(r_j) \qquad k = 0, 1, \dots, N_r$$

where T_k are the discrete Chebychev polynomials, b_k are the coefficients for the derivative and $-1 < r_j < 1$ is the j^{th} sampling point (Gottlieb and Orzag 1977, Kosloff et. al. 1989, Kessler and Kosloff 1990); Along the angular direction, we use the Fourier method for construction of the derivative (Hamming, 1978):

[5]

$$\frac{\partial}{\partial \theta}\tilde{F} = i\,K_\theta \tilde{F} \qquad \text{where} \quad K_\theta = j-1$$

and $j = 1, 2, \dots, N_\theta$

Along the vertical direction, we again use the Fourier method, where now:

[6]

$$\frac{\partial}{\partial z}\tilde{F} = i\,K_z \tilde{F} \qquad \text{with} \quad K_z = \frac{2\pi}{Nz \cdot Dz}(j-1)$$

and $j = 1, 2, \dots, N_z$

Integration in time is carried out using the fourth order Taylor series (Dablain, 1986):

[7]

$$U(t+\Delta t) = U(t) + \tilde{Q}\left[\tilde{M}U(t) + f\right]$$

$$\text{with} \quad \tilde{Q} = \left[\Delta t + \frac{\Delta t^2}{2}\tilde{M} + \frac{\Delta t^3}{6}\tilde{M}^2 + \frac{\Delta t^4}{24}\tilde{M}^3\right]$$

and $\tilde{M}$ is the right size of [3]; f is the source term, and Δt is the time step increment.

Boundary Conditions and **Domain Composition**

Boundary conditions are applied at the center of the grid and at the edges of the grid at the radial direction, and at the two edges of the grid in the vertical direction. At the angular direction, the operation is periodic.

Boundary conditions are applied by correcting the values of the nine component variables vector (equation [3]). The values of each component are defined by the use of the characteristic variables of the elastic wave equation (Gottlieb et. al., 1982, Baylis et. al., 1990, Kosloff et. al., 1990, Kessler and Kosloff, 1990). The nine characteristic variables are:

Downloaded 05 Jun 2012 to 72.20.129.98. Redistribution subject to SEG license or copyright; see Terms of Use at http://segdl.org/

$$\dot{U}_r + \frac{1}{\rho V_p}\sigma_{rr} \qquad \dot{U}_\theta - \frac{1}{\rho V_s}\sigma_{r\theta}$$
$$\dot{U}_\theta + \frac{1}{\rho V_s}\sigma_{r\theta}\ , \qquad \dot{U}_z - \frac{1}{\rho V_s}\sigma_{rz}$$
$$\dot{U}_z + \frac{1}{\rho V_s}\sigma_{rz}\ , \qquad \sigma_{rr} - \frac{\lambda+2\mu}{\lambda}\sigma_{\theta\theta}$$
$$\dot{U}_r - \frac{1}{\rho V_p}\sigma_{rr}\ , \qquad \sigma_{rr} - \frac{\lambda+2\mu}{\lambda}\sigma_{zz} \qquad [8]$$

and $\sigma_{\theta z}$

where V_p and V_s are respectively, the pressure and shear waves velocity. The above characteristic variables describes one sided and one dimensional wave propagation. In order to create the boundary condition needed, we keep constant or zero the terms of the variables which describes motion of energy inwards or outwards (Kessler and Kosloff, 1990).

The number of domains that we use are two. Figure 1 shows an (r,θ) section of the 3D numerical grid. The interior grid has less grid points in both the radial and the angular directions than the exterior one. Composition of the two domains is performed using the same technique we use for the performance of the boundary conditions, means by correcting the variables of the nine terms vector (equation [3]) using the characteristic variables of the wave equation (equation [8]).

Example and Computer Implementation

In this example we consider wave propagation in a homogeneous medium. The pressure wave velocity 2000 m/s and the shear wave velocity is 1300 m/s. The depth of the cylindrical structure is 1000 m. The number of grid points is $45 \times 45 \times 225$ in the interior grid and $45 \times 125 \times 225$ in the exterior grid. The total number of variables require for solving this problem reaches the number 59,000,000. A very powerful computer is needed for execution of this computer job. The computer we found most appropriate is the Cray YMP. The Cray YMP super computer possesses a very large physical memory and multiple CPUs which share the same memory. Figures 2 and 3 shows snapshots of the numerical solution. Figure 2 describes an horizontal (r,θ) cut of the radial velocity field at times 0.135 and 0.27 sec., respectively.

The white spot at the middle is the borehole, and it is surrounded by the interior grid, and then the exterior grid. Figure 3 shows a vertical (r,z) cut of the radial velocity field at times 0.28 and 0.42 sec., respectively. At these snap shots we can identify the pressure wave front followed by the shear wave front and also the waves partition along the borehole.

References

A. Baylis, K.E. Jordan, B.J. LeMesurier & E. Turkel, A forth-order accurate finite-difference scheme for the computation of elastic waves, Bulletin of the seismological society of america, Vol. 76, 1986.

M. A. Dablain, The application of high-order differencing to the scalar wave equation. Geophysics, Vol 51, 1986.

Y.C. Fung, Foundations of solid mechanics, Prentice-hall Inc., New Jersey, 1965.

Downloaded 05 Jun 2012 to 72.20.129.98. Redistribution subject to SEG license or copyright; see Terms of Use at http://segdl.org/

D. Gottlieb, M. D. Gunzberger and E. Turkel, On numerical boundary treatment for hyperbolic systems. SIAM J. Numer. Anal. Vol. 19, 1982.

D. Gottlieb and S. Orszag, Numerical analysis of spectral methods, theory and applications. Society For Industrial And Applied Mathematics, 1977.

R. Hamming, Numerical methods scientists and engineers, McGraw Hill book company, 1978.

D. Kessler and D. Kosloff, Acoustic wave propagation in 2-D cylindrical coordinates. Geophys. J. Int., Vol. 103, 1990.

D. Kosloff, D. Kessler, A. Q. Filho, E. Tessmer, A Behlle and R. Strahilevits, Solutions of the equations of dynamic elasticity by a Chebychev spectral method. Geophysics, Vol. 55, 1990.

Acknowledgements

We wish to express our appreciation to George Stevenson and Ted Clee of Cray Research Inc. for making available to us the resources of the Cray YMP super computer.

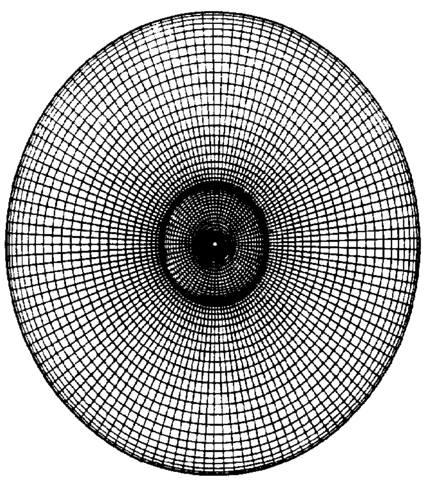

Fig. 1. Horizontal cut of the 3-D grid.

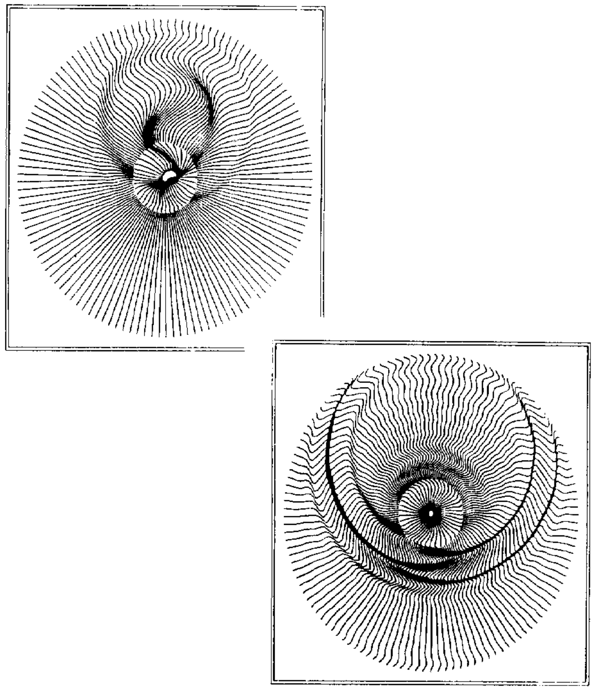

Fig. 2. (r,θ) snapshots at times 0.135 sec and 0.27 sec.

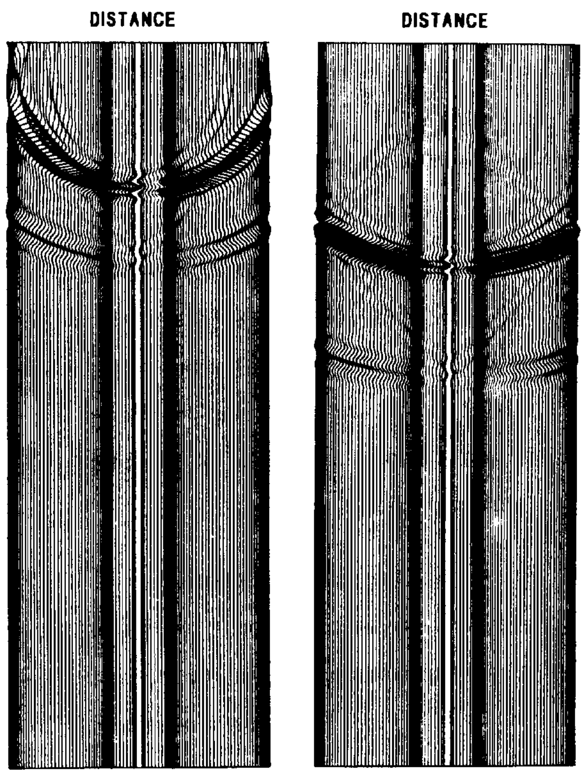

Fig. 3. (r,z) snapshots at times 0.28 sec and 0.42 sec.

Downloaded 05 Jun 2012 to 72.20.129.98. Redistribution subject to SEG license or copyright; see Terms of Use at http://segdl.org/