3-D Elastic Modeling by a Chebychev Spectral Method

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SUMMARY

The implementation of the free surface boundary condition into a composite spectral Chebyshev-Fourier method for solving the three dimensional equations of motion is presented. In this method spatial derivatives with respect to the vertical direction are calculated with a Chebyshev spectral method, while differencing with respect to the horizontal directions is performed with the Fourier method. The technique can handle free surface boundary conditions in a more rigorous manner than the ordinary Fourier method. It therefore appears well-suited for realistic full wave modeling, in particular of near surface layer problems.

Isotropic media and transversely isotropic media with a vertical axis of symmetry are considered. A comparison of the isotropic modeling results with the analytic solution for Lamb's problem shows the high accuracy of the alogorithm.

Modeling examples are presented for a 3D thin layer overlaying an elastic halfspace and a 3D transversely isotropic halfspace.

INTRODUCTION

In realistic seismic modeling the influence of the free surface cannot be ignored. Reflections and conversions at the free surface have to be taken into account as well as the influence of the low velocity weathering zone. The formulation of the free surface boundary condition is not always simple with grid methods. For low order finite differences the free surface condition can be applied with the same level of accuracy as the method itself (e.g. Vidale and Clayton, 1986). For forth order finite differences however only an approximation of the condition has been found (Bayliss et al., 1986). For the Fourier method free surface modeling requires "zero padding" (Kosloff et al., 1984). However, results are sufficiently accurate only if the source and the receivers are located at greater depth. While modeling the free surface is difficult with the ordinary Fourier method due to its periodicity, the non periodic Chebyshev expansion allows for a proper implementation of the free surface condition. In this way free surface situations can be correctly handled by the highly accurate spectral method.

EQUATIONS OF MOTION

The equations of motion for a three-dimensional isotopic medium are given by

$$\varrho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i, \quad i = 1,2,3 \tag{1}$$

where x_j are Cartesian coordinates, σ_{ij} are the stress components, u_i are the displacements, ϱ is the density and f_i denotes the body forces.

The stress-strain relations for an elastic isotropic medium are given by

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{2}$$

where λ and μ are Lame's parameters, δ_{ij} is Kronecker's symbol and ε_{ij} are the strains defined as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{3}$$

The material parameters ϱ, λ and μ can be arbitrarily vary in space. Equation (1) is a system of three coupled hyperbolic differential equations of second order in time. This system can be rewritten as a system of first order equations in time if we state it in terms of the particle velocity v_i and the stresses σ_{ij} instead of the displacements (e.g. Virieux, 1986; Bayliss et al., 1986).

We thus have

$$\begin{aligned}
\varrho \dot{v}_x &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x \\
\varrho \dot{v}_y &= \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y \\
\varrho \dot{v}_z &= \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z \\
\dot{\sigma}_{xx} &= \lambda\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) + 2\mu\frac{\partial v_x}{\partial x} \\
\dot{\sigma}_{yy} &= \lambda\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) + 2\mu\frac{\partial v_y}{\partial y} \\
\dot{\sigma}_{zz} &= \lambda\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) + 2\mu\frac{\partial v_z}{\partial z} \\
\dot{\sigma}_{xy} &= \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \\
\dot{\sigma}_{xz} &= \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \\
\dot{\sigma}_{yz} &= \mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)
\end{aligned} \tag{4}$$

With

$$\mathbf{W} = (v_x, v_y, v_z, \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz})^T$$

we can rewrite (4) as

$$\frac{\partial \mathbf{W}}{\partial t} = \mathbf{A}\frac{\partial \mathbf{W}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{W}}{\partial y} + \mathbf{C}\frac{\partial \mathbf{W}}{\partial z}, \tag{5}$$

where **A**, **B** and **C** are 9×9 matrices containing material parameters.

THE CHEBYCHEV SPECTRAL METHOD

The variables of (5) are discretized on a spatial grid which is uniform in the x and y directions and nonuniform in the z direction. The grid points in the z direction are calculated by a mapping $z_j = z_j(\xi_j)$ from the Chebyshev sampling points $\xi_j = \cos\frac{\pi j}{N}, j = 0, ..., N$ with $N+1$ grid points in the z direction. This mapping stretches the grid at the edges and as a consequence improves the stability requirements of the method. The mapping is discussed in detail in Kosloff and Tal-Ezer (1990). Horizontal derivatives are calculated by the Fourier method (Gazdag, 1981; Kosloff and Baysal, 1982). For the vertical derivatives a discrete Chebyshev expansion and the recursion relation for the coefficients of the derivative (Gottlieb and Orzag, 1977) have been used. The numerical implementation of this solution scheme has been described by Kosloff et al. (1990).

FREE SURFACE

The boundary condition for a flat free surface is zero traction, i.e.

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0. \quad (6)$$

The calculation of stresses by numerical methods normally gives non-zero values for σ_{xz}, σ_{yz} and σ_{zz} at the free surface. If we require zero values for these stresses, however, we have to correct for $\sigma_{xx}, \sigma_{yy}, v_x, v_y$ and v_z as well. Therefore we need a relation between v_i and σ_{ij} , which can be found by characteristic variables (Gottlieb et al., 1982; Bayliss et al., 1986).

A one-dimensional analysis (i.e. propagation of plane waves normal to the free surface) is used in order to calculate the characteristic variables. Thus we consider the following equation:

$$\frac{\partial \mathbf{W}}{\partial t} = \mathbf{C}\frac{\partial \mathbf{W}}{\partial z}. \quad (7)$$

After diagonalization an equation of the form

$$\frac{\partial \mathbf{S}}{\partial t} = \mathbf{\Lambda}\frac{\partial \mathbf{S}}{\partial z} \quad (8)$$

is found, where

$$\mathbf{\Lambda} = \mathbf{Q}^{-1}\mathbf{C}\mathbf{Q}$$

and

$$\mathbf{S} = \mathbf{Q}^{-1}\mathbf{W}.$$

$\mathbf{\Lambda}$ is a diagonal matrix and the components of the vector S are the characteristic variables. To calculate $\mathbf{Q}^{-1}$ the eigenvectors of C have to be found. These eigenvectors are column vectors of the matrix Q. From the inverse matrix $\mathbf{Q}^{-1}$ we can calculate the characteristic variables:

$$\mathbf{S} = \mathbf{Q}^{-1}\mathbf{W} = \frac{1}{2}\begin{pmatrix} 2\sigma_{xy} \\ 2\left(\sigma_{yy} - \frac{\lambda}{\lambda+2\mu}\sigma_{zz}\right) \\ 2\left(\sigma_{xx} - \frac{\lambda}{\lambda+2\mu}\sigma_{zz}\right) \\ \sigma_{yz} - \sqrt{\varrho\mu}\, v_y \\ \sigma_{xz} - \sqrt{\varrho\mu}\, v_x \\ \sigma_{yz} + \sqrt{\varrho\mu}\, v_y \\ \sigma_{xz} + \sqrt{\varrho\mu}\, v_x \\ \sigma_{zz} - \sqrt{\varrho(\lambda+2\mu)}\, v_z \\ \sigma_{zz} + \sqrt{\varrho(\lambda+2\mu)}\, v_z \end{pmatrix}. \quad (9)$$

The specification of the boundary conditions at the free surface, i.e. $\sigma^N_{xz} = \sigma^N_{yz} = \sigma^N_{zz} = 0$ yields together with the condition that outgoing characteristic variables remain unmodified after application of the boundary conditions (Gottlieb et al., 1982):

$$\begin{aligned} \sigma^N_{yy} &= \sigma_{yy} - \frac{\lambda}{\lambda+2\mu}\sigma_{zz} \\ \sigma^N_{xx} &= \sigma_{xx} - \frac{\lambda}{\lambda+2\mu}\sigma_{zz} \\ v^N_y &= v_y + \frac{\sigma_{yz}}{\sqrt{\varrho\mu}} \\ v^N_x &= v_x + \frac{\sigma_{xz}}{\sqrt{\varrho\mu}} \\ v^N_z &= v_z + \frac{\sigma_{zz}}{\sqrt{\varrho(\lambda+2\mu)}} \end{aligned} \quad (10)$$

The superscript N denotes the values of the variables at the free surface after the correction. These corrections have to be applied to stresses and particle velocities. The first characteristic variable, i.e. σ_{xy}, remains unchanged.

EXAMPLES

Fig. 1 shows the geometry of the 3D isotropic model. A thin layer of 15 m thickness is overlaying a homogeneous halfspace. The velocity of the P- and S-waves are 2000 m/s and 1155 m/s in the layer and 4000 m/s and 2309 m/s in the halfspace, respectively. The model is discretized with 125 × 125 × 81 gridpoints in the x-, y- and z-direction, respectively. The grid spacing is 10 m.

In this example a horizontal point force in x-direction with 50 Hz cutoff frequency is used. Snapshots of the x- and the z-component of the particle velocity on the surface at 500 ms propagation time are shown in Fig. 2. Since the source was placed only 0.9 m below the surface the energy is mainly propagating as surface waves. In case of vertical inhomogeneous media and a horizontal point force Love- and Rayleigh-waves are generated. Both waves show dispersion. Besides the body waves in the x-direction only Rayleigh-waves and in the y-direction only Love-waves are propagated.

Fig. 3 shows snapshots of vertical planes at 500 ms propagation time. In the xz-plane (top) the Rayleigh wave and the S-type body wave can be seen, while the yz-plane (bottom) shows the Love-wave and also the S-wave.

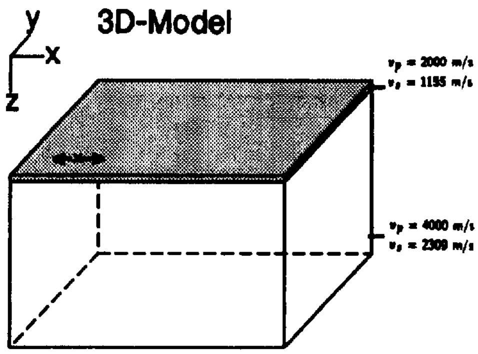

Fig. 1

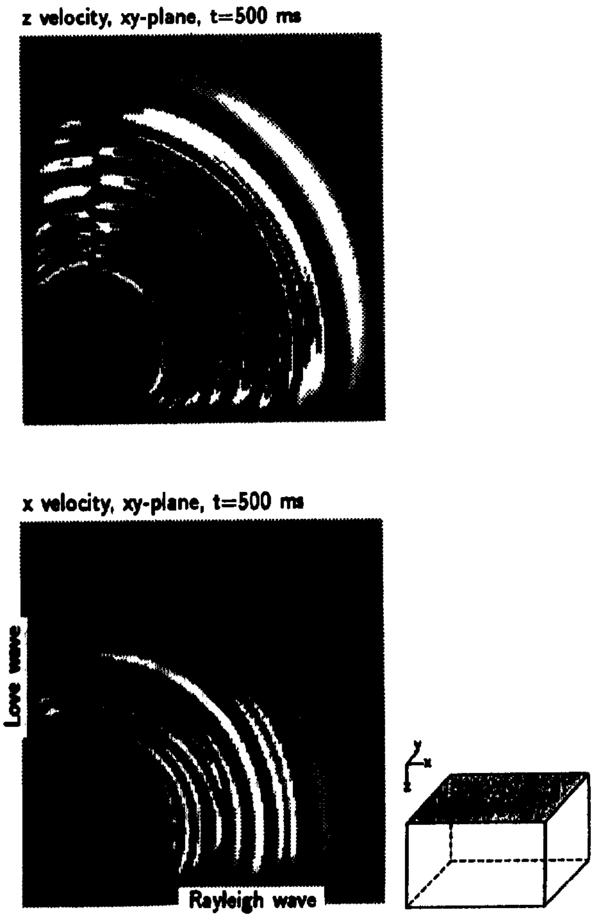

Fig. 2

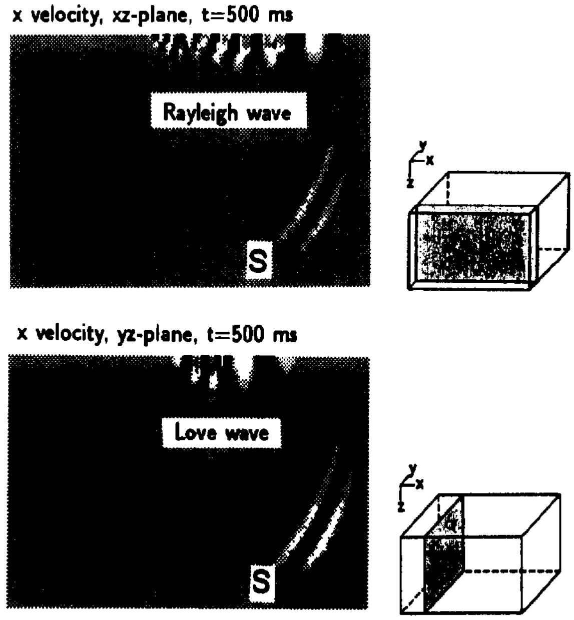

Fig. 3

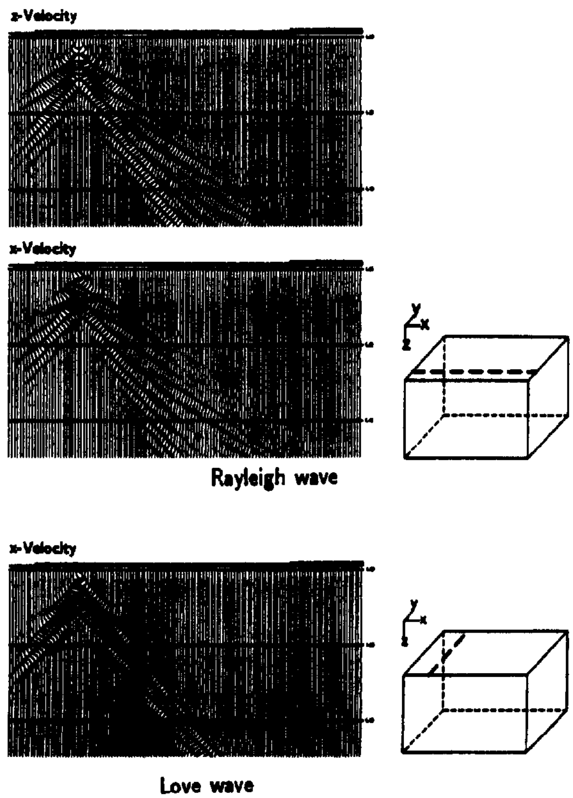

Fig. 4

Seismogram recordings in the x- and z-directions at lines indicated by dashed lines also show the surface wave arrivals (Fig. 4). The dispersive character of the surface waves can be seen very clearly.

An example of wave propagation in a 3D transversely isotropic halfspace is shown in Fig. 5. The model is discretized by 135×81×81 gridpoints in the x-, y- and z-direction, respectively. The gridspacing is 8 m.

The qP-wave velocity in the medium varies between 2740 m/s and 2960 m/s. The qSV velocity is 860 m/s up to 1415 m/s, while the qSH velocity varies between 1000 m/s and 1415 m/s.

The snapshots show the vertical (z) component of the particle velocity in a xz-plane at 200 and 400 ms propagation time. The horizontal point force was placed at a depth of 0.75 m. Both snapshots show the wave front of the qP-wave and the qSV-wave. The later snapshot (400 ms) shows the qSV-wave with the cusps and a headwave generated at the free surface.

A comparison of the modeling results with the analytical solution of Lamb's problem in an isotropic half space shows very good agreement (Fig. 6). The source with a 50 Hz cutoff frequency Ricker wavelet is placed at 106 m depth. The horizontal offset is 60 m. Since the source is a vertical point force the problem is symmetric about the vertical axis. The figure shows comparisons of the horizontal (x) and the vertical (z) displacements.

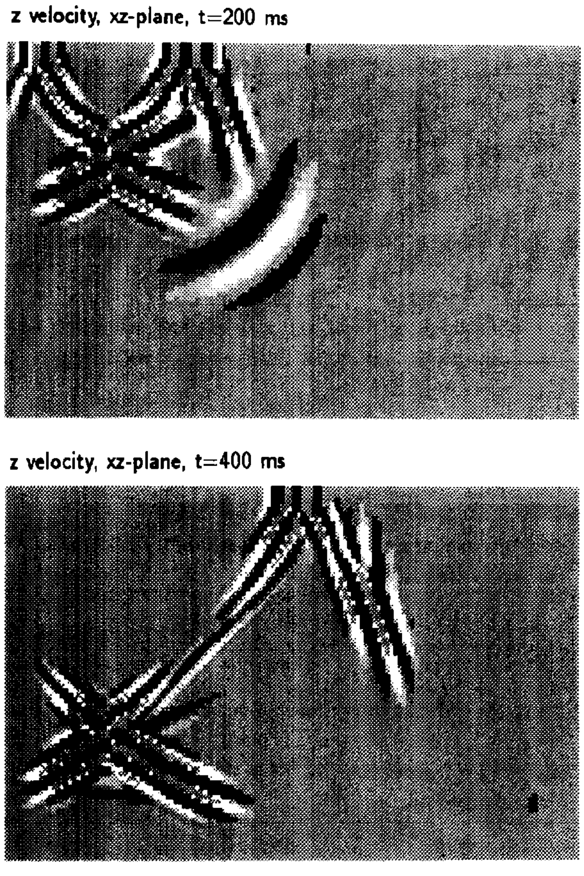

Fig. 5

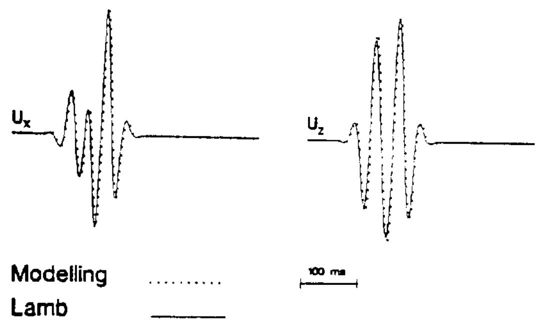

Fig. 6

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