Three-Dimensional Solutions of Equations of Dynamic Elasticity by a New Rapid Expansion Method (REM)

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SUMMARY

This work represents a new scheme for time integration of direct solution methods such as finite-difference, finite-element, and Fourier methods. Forward modeling with elastic wave propagation is investigated. The rapid expansion method (REM), which we have developed, is based on a modified Chebychev expansion of the formal solution to the governing equations. The REM implementation of the equations of dynamic elasticity using the Fourier method is similar to previous development featuring conventional time integration of second order differencing. A high degree of parallelism exists between the calculations in one time step with temporal differencing and REM calculations. Spatial partial derivatives are computed in the same manner. Compared to time integration of second order differencing, larger time steps or increments are permissible with the REM implementation. For output time sections, results at intermediate times are obtained by resubstituting intermediate times in the computational equations. This does not require recomputation of spatial partial derivatives. The REM can be used to increase accuracy or to obtain comparable accuracy with fewer computations. Conventional time integration and REM methods have been implemented on the CRAY X-MP computer system using parallel processing and the large memory of the Solid-state Storage Device (SSD). The REM can be applied to acoustic and elastic wave propagation using formulations other than the Fourier method.

INTRODUCTION

For acoustic and elastic wave propagation, direct solution methods typically feature distinct temporal and spatial differentiation approximations. Spatial approximations are accomplished by a number of approaches with varying degrees of accuracy. Temporal derivatives are typically approximated by some form of differencing based on Taylor expansions. For algorithms with highly accurate spatial approximations, the above has caused an imbalance between temporal and spatial accuracy.

BASIC EQUATIONS FOR AN ISOTROPIC ELASTIC SOLID

The numerical algorithm is based on the solution of the equations of momentum conservation for a three-dimensional continuous medium combined with the strain-displacement and stress-strain relations for a linear isotropic elastic solid (Fung, 1965).

Written in compact tensor notation, the equations of momentum conservation are the following,

$$\rho \ddot{u}_i = \frac{\partial}{\partial x_j} \sigma_{ij} + \rho f_i \qquad i = 1,2,3, \tag{1}$$

where x_j are Cartesian coordinates, $\sigma_{ij}(\vec{x},t)$ are the stresses, $\rho(\vec{x})$ denotes the density, $u_i(\vec{x},t)$ denote displacements, and $f_i(\vec{x},t)$ are the body forces. In (1) and elsewhere, repeated indices imply summation and time differentiation is denoted by the dot convention.

The stress-strain relations are given by,

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \qquad i,j = 1,2,3, \tag{2}$$

where $\lambda(\vec{x})$ and $\mu(\vec{x})$ are the rigidity and shear modulus, respectively, $e_{ij}(\vec{x},t)$ are the strains, and δ_{ij} is the unit tensor or Kronecker delta.

Finally, the strain-displacement relations are given by,

$$e_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} u_i + \frac{\partial}{\partial x_i} u_j\right) \qquad i,j = 1,2,3. \tag{3}$$

FORMAL SOLUTIONS

After spatial discretization, direct methods yield a coupled system of differential equations for the displacements at the nodes of the numerical mesh. This system can be written in compact form as,

$$\frac{\partial^2}{\partial t^2} \vec{u} = -L^2\vec{u} + \vec{f}, \tag{4}$$

where $\vec{u}$ is a vector containing the displacement in x, y, and z directions, $\vec{f}$ is a vector of body forces, and $-L^2$ is a linear operator.

Assuming that body forces are separable according to $f(\vec{x},t)=g(\vec{x})h(t)$ where all time dependence is contained in h(t), the formal solution to (4) under zero initial conditions is given by,

$$\bar{u}(t) = \left[\int_o^t \frac{\sin L\tau}{L} h(t-\tau)d\tau \right] \bar{g}. \tag{5}$$

Conversely, when the body forces are zero and with initial conditions $\vec{u}(o)$ and $\vec{\dot{u}}(o)$ for the displacements and first time derivatives, respectively, the solution to (4) becomes the following,

$$\bar{u}(t) = \cos Lt\, \bar{u}(o) + \frac{\sin Lt}{L} \bar{\dot{u}}(o). \tag{6}$$

REM FOR THE NON-ZERO BODY FORCES

Following the work of Tal-Ezer (1986), the REM is based on the following Chebychev expansion,

$$\frac{\sin L\tau}{L} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{J_k(\tau R)}{R} \left(\frac{R}{iL} \right) Q_k \left(\frac{iL}{R} \right) \tag{7}$$

where R is a scalar larger than the range of eigenvalues of L, J_k denotes the kth order Bessel function, and Q_k are modified Chebychev polynomials, satisfying the following recursion relation,

$$\begin{aligned} &\frac{R}{iL} Q_1 \left(\frac{iL}{R} \right) = I, \\ &\frac{R}{iL} Q_3 \left(\frac{iL}{R} \right) = 3I - \frac{4L^2}{R^2}, \\ &\frac{R}{iL} Q_{k+2} \left(\frac{iL}{R} \right) = \left(\frac{-4L^2}{R^2} + 2I \right) \frac{R}{iL} Q_k \left(\frac{iL}{R} \right) \\ &\qquad - \frac{R}{iL} Q_{k-2} \left(\frac{iL}{R} \right), \end{aligned} \tag{8}$$

where I is the identity operator. Using (7), the solution (5) can be written as follows,

$$\vec{u}(t) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} b_k \left(\frac{R}{iL} \right) Q_k \left(\frac{iL}{R} \right) \bar{g}, \tag{9}$$

where

$$b_k = \frac{1}{R} \int_o^t J_k(\tau R) h(t-\tau) d\tau. \tag{10}$$

For a band limited time history h(t), b_k terms decay exponentially and summation (9) is truncated after approximately R/2 terms. b_k terms are evaluated by numerical integration.

REM FOR ZERO BODY FORCES

To eliminate displacement time derivatives, we add solutions to (6) for times t and -t obtaining the following,

$$\bar{u}(t) = -\bar{u}(-t) + 2 \cos Lt\, \bar{u}(o). \tag{11}$$

The REM is obtained with the following expansion,

$$\cos Lt = \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} c_k J_k(tR)\, Q_k \left(\frac{iL}{R} \right), \tag{12}$$

where $c_o=1$ and $c_k=2$ for $k\neq0$ (Tal-Ezer, 1986). Q_k satisfy the following recursion relation,

$$\begin{aligned} &Q_o \left(\frac{iL}{R} \right) = I, \\ &Q_2 \left(\frac{iL}{R} \right) = I - \frac{2L^2}{R^2}, \\ &Q_{k+2} \left(\frac{iL}{R} \right) = \left(\frac{-4L^2}{R^2} + 2I \right) Q_k \left(\frac{iL}{R} \right) - Q_{k-2} \left(\frac{iL}{R} \right). \end{aligned} \tag{13}$$

Summation (12) is truncated after approximately R/2 terms. Given values of $\vec{u}(-t)$ and $\bar{u}(o)$, the solution can be propagated to time t by (11-13).

REM IMPLEMENTATION TO THREE-DIMENSIONAL ELASTICITY

The three-dimensional elastic implementation with the Fourier method requires twelve global arrays which reside on SSD storage. These are material parameters ρ, λ, μ, three displacement components, three Q_{k-2} terms, and three Q_k terms. Three additional SSD arrays are required for temporary storage. This is the same number of arrays required for implementation with temporal differencing (Edwards et al., 1985, and Reshef et al., 1987). Data motion and calculation of spatial derivatives are also identical to the previous implementation.

For a specified model, calculations first proceed from zero initial conditions with the non-zero body force formulation. After source termination, the solution is propagated by the zero body force formulation. It is efficient to use fairly large time increments (for example, 100-200 msec). As implemented, central memory storage of expansion terms, which increases with time increment size, is the limiting factor. For output time sections, results at intermediate times are obtained from (9-10) or (11-12) by substituting the intermediate time instead of t. This does not involve additional calculation of spatial derivatives.

ACCURACY AND NUMERICAL EFFICIENCY

Expansions (7) and (12) can be calculated with machine accuracy at no significant added cost. For practical purposes, no errors are introduced by the REM in the time integration. This is unlike temporal differencing where numerical dispersion is always present (Kosloff and Baysal, 1982). The main source of error is due to the representation of a continuous medium by a discrete spatial mesh. This factor will always limit the highest frequency which can be correctly propagated.

We will now consider the efficiency of REM and temporal differencing with the Fourier method. For comparison, we examine the number of $-L^2$ operator evaluations required to propagate the solution 100 grid lengths for a medium with uniform P-wave velocity V_p. For the REM method, the number of expansion terms (or, equivalently, the number of $-L^2$ operator evaluations) is approximately 20% greater than $\frac{1}{2}R\Delta t$ where Δt is the total propagation time. For three-dimensional propagation, R is approximately equal to $(\sqrt{3} * V_p)/\Delta s$ and the propagation time is $(100\Delta s)/V_p$ where Δs is the grid spacing. Approximately 326 expansion terms are required. For second order temporal differencing, the parameter $\alpha=(V_p\Delta t)/\Delta s$ must be kept smaller than 0.2. For propagation of 100 grid lengths, at least 500 $-L^2$ operator evaluations or time steps are required. As most of the computational effort is in the evaluation of the $-L^2$ operator, the ratio between the total computation times with the two methods is proportional to the respective number of required operator evaluations. Numerical examples for three-dimensional elastic wave propagation are presented.

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